Propagation of solutions of the Porous Medium Equation with reaction and their travelling wave behaviour.

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13 March 2018

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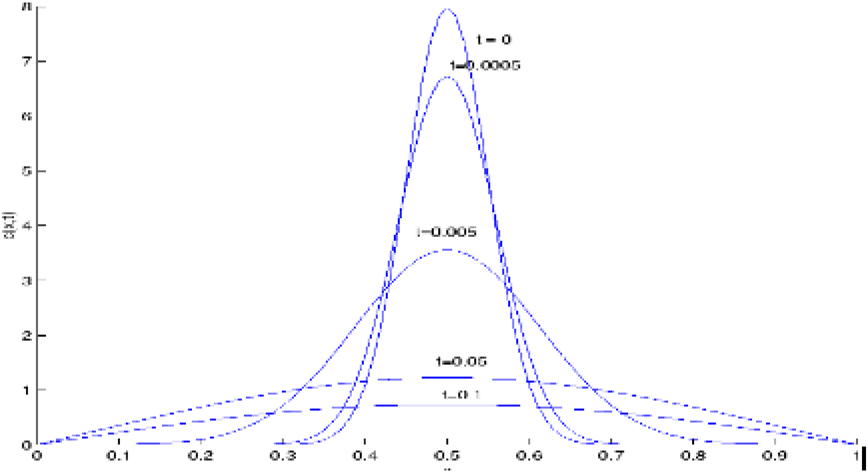
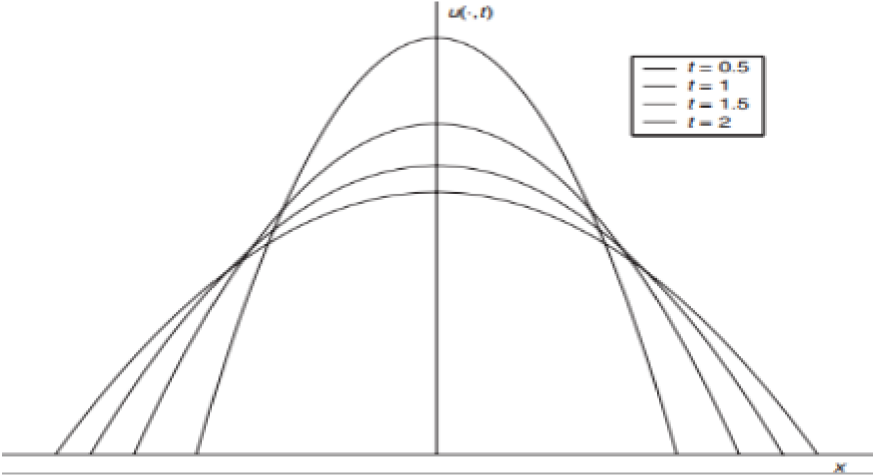
The Heat Equation.

$$u_t = \Delta u$$



Porous Medium Equation.

$$u_t = \Delta u^m = \Delta(D(u)\cdot u), \quad D(u) = m u^{m-1}$$



Physical motivations.

- **Biology** - Growth of population depending on its density and a Pearl-Verhaulst type reaction.

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- *Newman, W. I.; Sagan, C. Galactic civilizations: population dynamics and interstellar diffusion. Icarus 46 (1981), 293–327.*

The model.

For $m > 1$

$$\begin{cases} u_t = \Delta u^m + h(u) & \text{in } Q := \mathbb{R}^N \times \mathbb{R}_+, \\ u(\cdot, 0) = u_0 \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

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The reaction term h is assumed to be in $C^1(\mathbb{R}_+)$ and to fulfill, for some $a \in [0,1)$,

$$\begin{cases} h(0) = 0, \quad h'(1) < 0 \\ h(u) \leq 0 \text{ if } u \in [0, a], \\ h(u) > 0 \text{ if } u \in (a, 1), \\ h(u) < 0 \text{ if } u > 1, \\ \\ \end{cases}$$

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Arising questions.

- When does the solution u spread the value 1 along the medium?
- How fast does it propagate?
- Which shape does the solution take when propagating?

Key tool. The travelling waves.

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The unique travelling wave solution.

If our solution propagates, it probably is similar to a travelling wave solution, a function $V(x-ct) \equiv V(\xi)$ that satisfies (1), i.e.,

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Theorem (Gilding, Kersner).

There exists a minimal speed $c^* = c^*(m,h) > 0$ such that equation (1) has an unique (up to translations) distinct monotonic "change of fase type" TW solution satisfying

$$\lim_{\xi\to-\infty} V_{c^*}(\xi) = 1, \qquad \lim_{\xi\to\infty} V_{c^*}(\xi) = 0.$$

and this TW with speed c^* is $0 \le V_{c^*} < 1$ and finite.

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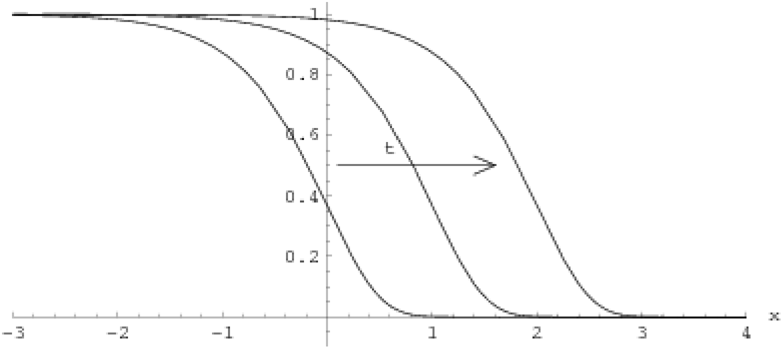

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For a special pair of functions h^* and ϕ we consider the following system of ODEs:

$$\begin{cases} f'(t) = h^*(f), \quad f(0) = f_0 > a, \\ g'(t) = c^*\phi(f) - h^*(f)/f, \quad g(0) = g_0, \end{cases}$$

and functions $w(x,t) = f(t)V(x - g(t))$. Let δ be such that $h'(u) < 0$ for all $u \in (1-\delta, 1+\delta)$

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In both cases, $w(\xi + c^*t, t) \to V(\xi - \xi_0)$ and $(g(t) - c^*t) \to \xi_0$ when the time t goes to infinity.

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We focus for now on initial data that are non-negative, bounded, piece-wise continuous and compactly supported. We will reffer to this as compactly supported initial data.

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Our first result states that for every reaction h there exist certain initial data of compact support for which our solution propagates. It depends on how much mass u_0 has and how concentrated it is.

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Theorem 1.1 (G., Quirós).

There exists a three-parameter (x_0, η, ρ) family of functions v such that if

$$u(x,0) \geq v(x; x, \eta, \rho)$$

for some $x_0 \in \mathbb{R}^N, \eta > 0$ and $\rho > 0$, then u converges to 1 uniformly on compact sets.

There can always be spreading for every initial datum.

Next, we see that certain reactions always lead to propagation, regardless on the mass of the initial datum. It depends on the behaviour that h presents near $u = 0$ compared to the **Fujita exponent**. This is called the **hair-trigger effect**.

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Theorem 1.2 (G., Quirós).

Suppose that

$$\liminf_{u \to 0} \frac{h(u)}{u^{m+2/N}} > 0.$$

and that $u \neq 0$.
Then u converges to 1 uniformly on compact sets.

Fujita exponent for the PME in dimension N:

$$p_F = m + 2/N$$

Speed of propagation

If we move from a point $y_0 \in \mathbb{R}^N$ in a certain direction with a speed c, what will we see as time grows, an empty enviroment or a saturated one? It depends on said speed.

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Theorem 1.3 (G., Quirós).

Whenever spreading happens, for any $c \in (0, c^*)$

$$\lim_{t\to\infty} \min_{|y-y_0|\le ct} u(y,t) = 1.$$

and for any $c > c^*$

$$\lim_{t\to\infty} u(y,t) = 0 \quad \text{for} \quad |y - y_0| \ge ct.$$

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Too slow will translate to $c < c^*$ (saturated enviroment), and too fast to $c > c^*$ (empty enviroment). In this sense, the speed c^* will be called the critical speed of the problem.

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We turn our attention to initial data that are non-negative, bounded, piecewise continuous, $u_0(x) \equiv 0$ for all $x \geq x_0,\ x_0 \in \mathbb{R}$ and

$$\liminf_{x \to -\infty} u_0(x) > \delta,$$

where δ is such that $h'(u) < 0$ for all $u \in (1-\delta, 1+\delta)$. This class of functions will be called $\mathcal{A}$ class.

Convergencia uniforme en todo el espacio

We can achieve convergence in all $\mathbb{R}$ to a travelling wave V solution of (3) with speed c^*.

Theorem 2 (G., Quirós).

Let the dimension be $N = 1$. Let u be a solution to (1) with $u_0 \in \mathcal{A}$. Then there exists a $\xi_0 \in \mathbb{R}$ such that

$$\begin{aligned} &\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - V(x - c^*t - \xi_0)| = 0, \\ &\lim_{t\to\infty} \zeta(t) - c^*t = \xi_0. \end{aligned} \tag{4}$$

Here $\zeta(t)$ denotes the free boundary of the solution.

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5) If $u(\xi,t)$ oscilates, the Ascoli-Arzelá theorem and a bit of extra work give convergence along a certain time sequence without degeneration, perhaps different from the one in step 2.
6) We use a stability result, assuring that if our solution is close to a profile V in a certain time, it will remain close for all posterior times, giving convergence along all time sequences.

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> Can we get to uniform convergence in dimension 1 for compactly supported initial data?

Keep in mind that the results about the critical speed c^* don't say anything about the shape of the function for long times near the front, i.e., near the free boundary.

Convergence of solutions of compact support in $\mathbb{R}$.

We can prove, as expected, uniform convergence via a proof similar to the one for the $\mathcal{A}$ class. We define $\Omega_1 \equiv [0, \infty)$ and $\Omega_2 \equiv (-\infty, 0]$.

Theorem 3 (G., Quirós).

Let the dimension be $N = 1$ and u be a solution of equation (1) of compact support that converges to 1 uniformly in compact sets. Let W be the reflexion of V, i.e., $V(\xi) = W(-\xi)$ and V_ξ, W_ξ translations of said profiles. Then there exist a pair of values ξ_* and ξ^* such that

$$\lim_{t\to\infty} |u(x,t) - V_{\xi^*}(x,t)| = 0 \quad \text{in} \quad \Omega_1, \tag{5}$$

$$\lim_{t\to\infty} |u(x,t) - W_{\xi_*}(x,t)| = 0 \quad \text{in} \quad \Omega_2. \tag{6}$$

Convergence in greater dimensions.

For radially symmetric initial data of compact support, we expect a Branson correction to appear, i.e.,

$$\lim_{t\to\infty} |u(x,t) - V_{\xi^*}(x,t) + (N-1)c^* \log(t)| = 0.$$

Losely speaking, it comes from the formula for the laplacian in radial coordinates, which transforms our equation in

$$u_t = (u^m)_{rr} + \frac{N-1}{r}(u^m)_r + h(u).$$

Bibliography

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